Correlation - Modelling

Viral Acharya & Stephen M Schaefer
NYU-Stern and London Business School, London Business School

Credit Risk Elective
Spring 2009
Valuation of CDO Tranches
Tranched 125 Name DJ.CDX.NA.IG Series 5
(Illustrative Pricing 16 Feb, 2006)

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Estimated Rating</th>
<th>Market Quote (bp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15% - 30%</td>
<td>AAA (junior super senior)</td>
<td>4/5</td>
</tr>
<tr>
<td>10% - 15%</td>
<td>AAA (junior super senior)</td>
<td>12/13</td>
</tr>
<tr>
<td>7% - 10%</td>
<td>AAA (junior super senior)</td>
<td>26/26</td>
</tr>
<tr>
<td>3% - 7%</td>
<td>BBB-</td>
<td>108/110</td>
</tr>
<tr>
<td>0% - 3%</td>
<td>Not rated</td>
<td>35.4% / 35.9% + 500 bp</td>
</tr>
</tbody>
</table>

Source: Morgan Stanley
Attachment and Detachment Points

• Each tranche is defined in terms of its *attachment* ($\beta_A$) and *detachment* ($\beta_D$) points.
  ✓ these are measured in terms of losses as percent of total face value of basket.

• The *attachment* point defines the limit *below which* the tranche bears *none* of the *loss*.

• The *detachment* point defines the limit *above which* the tranche loss *does not increase*.
Tranche Loss Payments

- If total losses (as a percent of the total nominal portfolio value) are $L$, then for a tranche with attachment and detachment points $\beta_A$ and $\beta_D$ the tranche loss payment is:

$$
\text{Tranche Loss} = \begin{cases} 
0 & L < \beta_A \\
L - \beta_A & \beta_A \leq L \leq \beta_D \\
\beta_D - \beta_A & L > \beta_D
\end{cases}
$$
Tranche Loss Payments: Equity and Senior Tranche

• **Equity tranche** loss is **concave** in portfolio loss: expected loss on tranche **decreases** (and **value of tranche increases**) with variance of portfolio loss

• **Senior tranche** loss is **convex** in portfolio loss: expected loss on tranche **increases** (and **value of tranche decreases**) with variance of portfolio loss
Effect of Correlation on Loss Distribution and Tranche Values

<table>
<thead>
<tr>
<th>Attachment</th>
<th>Equity</th>
<th>Mezz</th>
<th>Senior</th>
<th>Total*</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>0%</td>
<td>3%</td>
<td>7%</td>
<td>0%</td>
</tr>
<tr>
<td>3%</td>
<td>3%</td>
<td>7%</td>
<td>100%</td>
<td>100%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Expected Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>2.94% 2.39% 0.72% 6.0%</td>
</tr>
<tr>
<td>25%</td>
<td>2.29% 1.64% 2.12% 6.0%</td>
</tr>
</tbody>
</table>

Portfolio Loan Loss

High Corr: $p = 6.0\%$, $\rho = 25.0\%$
Low Corr: $p = 6.0\%$, $\rho = 5.0\%$
Valuation Method: the Idea

• **Three** steps:
  ✓ valuation of the *protection* leg
  ✓ valuation of the *premium* leg
  ✓ Calculation of tranche *running spread* as the value that equates the value of the premium and protection legs.
Protection Leg Valuation

- Need to calculate distribution of portfolio losses:
  - Suppose portfolio contains 125 names
  - Use Monte Carlo and (e.g.) Gaussian copula to compute drawings from the joint distribution of default times for 125 names
  - For each name \( (k) \) and each trial \( (j) \) in simulation, if simulated default time is \( \tau_{k,j} \) calculate default event indicator \( \text{Def}_{k,j} \):

\[
\text{Def}_{k,j} = \begin{cases} 
0 & \text{no default} \\
1 & \text{default} 
\end{cases} \quad \tau_{k,j} \text{ Maturity} \quad \tau_{k,j} < \text{Maturity}
\]
Protection Leg Valuation: Portfolio Loss Percentage

Now, for each trial, \( j \), count the number of defaults and divide by 125 to give the loss frequency (as a percentage) and then multiply by \( LGD \) to give the portfolio loss as a percentage of total portfolio face value:

\[
L_{Port,j} = LGD \sum_{k=1}^{125} \frac{Def_{k,j}}{125} = LGD \quad \text{Loss Frequency (\%)}
\]
Tranche Loss Percentage

- Using the relation between the tranche loss and the portfolio loss, compute the *tranche loss* \( TL(L_{\text{port},j}) \) for this trial.
- In other words, for tranche \( m \) and trial \( j \) compute:

\[
TL_{m,j} = T_m(L_{\text{Port},j})
\]

- Where the function \( T_m(L_{\text{port},j}) \) – *shown at right for the 3%-7% tranche* – gives the tranche loss as a function of the total portfolio loss for tranche \( m \) (both as a percent of total portfolio face value).
Value of Protection Leg

To calculate the value of the protection leg $V_{m,Prot}$ (again as a percentage of the portfolio face value) we now simply calculate the discounted average value of the tranche loss over the $N$ Monte-Carlo trials and discount this to the present*:

$$V_{m,Prot} = e^{-rT} E(TL_{m,j})$$

$$= e^{-rT} \frac{1}{N} \sum_{j=1}^{N} TL_{m,j}$$

*Note: this assumes that all loss payments are made at maturity, $T$. 
Calculating Tranche Losses from Total Portfolio Loss

- The calculation of tranche losses is illustrated in the three tranche example below:
  - from the portfolio loss the losses attributed to the three tranches are calculated by reference to the attachment and detachment points
  - the *expected value* of each tranche loss (previous slide) is then just the average of the column of losses for that tranche

<table>
<thead>
<tr>
<th>Trial (j)</th>
<th>Portfolio Loss</th>
<th>Equity (0% - 3%)</th>
<th>Mezz (3%-7%)</th>
<th>Senior (7%-100%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5%</td>
<td>3%</td>
<td>2%</td>
<td>0%</td>
</tr>
<tr>
<td>2</td>
<td>1%</td>
<td>1%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>3</td>
<td>2%</td>
<td>2%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>4</td>
<td>10%</td>
<td>3%</td>
<td>4%</td>
<td>3%</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Premium Leg Valuation

• We **assume** that **full premium** is received until **end of contract irrespective** of number of defaults.

  ✓ in fact, if one (say) of the 125 names defaults, the premium received across all the tranches will be the reduced by $1/125^{th}$ from that time.

• Ignoring this, the present value of the premium leg is simply the **value of an annuity** paying the spread, $S$ up to contract maturity

• This is expressed as a **percentage of the total exposure** of the tranche which is the face value, $F$, multiplied by $(\beta^D - \beta^A)$, the **width of the tranche**.

\[
P_{\text{Prem}} = S \left( \beta^D - \beta^A \right) F \frac{1}{\text{tranche exposure}} \left( 1 - \exp(-rT) \right) \frac{1}{\text{annuity factor}}
\]
Calculating the Running Spread

• Finally, we equate the $PV$ of the premium leg (expressed as a percentage of the face value of the portfolio) to the value of the protection leg:

\[
V_{m,\text{Prot}} = e^{-rT} \frac{1}{N} \sum_{j=1}^{N} TL_{m,j} = S \left( \beta_D - \beta_A \right) \frac{1}{r} \left( 1 - \exp(-rT) \right)
\]

• And so the running spread is given by:

\[
S = \frac{e^{-rT} \frac{1}{N} \sum_{j=1}^{N} TL_{m,j}}{\left( \beta_D - \beta_A \right) \frac{1}{r} \left( 1 - \exp(-rT) \right)}
\]
CDS Spreads on Indices and Tranches

CDS index spreads

| Master investment grade indices | Tranches
| North America | Europe | 3-7% | 7-10% | 10-15% | 15-30% |
|---|---|---|---|---|---|---|
| Jan 04 | Mar 04 | May 04 | Jul 04 | Sep 04 | Nov 03 | Feb 04 | May 04 | Aug 04 |
| 60 | 40 | 20 | 0 | 450 | 300 | 150 | 0 |

1 On-the-run five-year swap spreads, in basis points.

2 North America master investment grade.

Source: JPMorgan Chase.

Source: Amato and Gyntelberg, “CDS Index Tranches and the Pricing of Credit Risk Correlations”, BIS Quarterly Review, March 2005
Correlation
Tranche Spreads and Correlation

Note: number of names = 100; CDS spread =100 bps; LGD = 0.6;
Dependence of Value of CDX Tranches on Correlation (Gaussian Copula)

<table>
<thead>
<tr>
<th>Price sensitivity of CDX tranches to default time correlation(^1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–3% tranche(^2)</td>
</tr>
<tr>
<td><img src="graph1.png" alt="Graph 1" /></td>
</tr>
</tbody>
</table>

\(^1\) Based on correlation sensitivities reported in Hull and White (2004).
\(^2\) Upfront payment, in per cent.
\(^3\) Spread, in basis points.

Sources: Hull and White (2004); BIS calculations.

Source: Amato and Gyntelberg, “CDS Index Tranches and the Pricing of Credit Risk Correlations”, BIS Quarterly Review, March 2005

Acharya & Schaefer: Copula approach to valuing CDO tranches
Implied Correlation 1/2

• In applications of the Black-Scholes model, there is most disagreement/uncertainty about the volatility parameter
  ✓ Traders use implied volatility (volatility calculated from the option price) as a measure of the relative value of the option

• In the same way, for CDO tranches there is most disagreement/uncertainty about the pattern of correlation

• Implied correlation for CDO tranches is calculated in two main ways.
Implied Correlation 2/2

• *Implied* correlation
  ✓ tranche specific correlation is the value that is consistent with the market spread of each tranche (*one at a time*)

• *Base* correlation
  ✓ Consider a structure with detachment points of 3%, 10%, 30% etc.
  ✓ The base correlation for the 10% detachment point is the correlation that is consistent with the *combined market values* of the 0%-3% and 3%-10% tranches.
Implied and Base Correlation

Note: the x-axis is tranche (left-hand panel) and tranche size (right-hand panel).

1 Based on on-the-run five-year contracts. 2 Averages based on correlation sensitivities reported in Hull and White (2004) and market quotes from JPMorgan Chase spanning the period 13 November 2003 to 28 September 2004.

Sources: Hull and White (2004); JPMorgan Chase; BIS calculations.

Source: Amato and Gyntelberg, “CDS Index Tranches and the Pricing of Credit Risk Correlations”, BIS Quarterly Review, March 2005
Some Perspective on Correlation

• Need to remember that implied correlation is a single statistic that is supposed to summarise 7750 different correlation parameters in a basket of 125 names (125x124/2)

• CDO value (and therefore correlation) coefficient also depends strongly on the type of copula being used
  ✓ we have used Gaussian copula – market standard for “communication” between market practitioners – but no empirical evidence that this is a good model.

• Pattern of default correlation major area for future work
Mark-to-Market (MTM) Impact of Spread Change

• Suppose protection *seller* via a particular tranche *receives* a spread of $s_0$

• Now suppose that market spread increases to $s_1$:
  ✓ if seller wishes to close out position s/he will have to purchase protection at $s_1$, leading to a net *outflow* of $(s_1 - s_0)$
  ✓ the *MTM impact* of this change is the *PV* of the change in the spread, i.e., the value of an *annuity* paying $(s_1 - s_0)$
  ✓ *value* to seller *increases* for *reduction in spread*
Delta and MTM Hedging

• Suppose we wish to *hedge a CDO* against changes in the CDS spreads of the underlying names

• The *delta* of a CDO measures the sensitivity of the value of a tranche to a change in the value of a particular given underlying CDS

\[
\text{Delta of Credit} = \frac{\text{Change in Mark-to-Market of Tranche}}{\text{Change in Mark-to-Market of Credit}}
\]

• Delta usually expressed as percentage of CDS position in underlying portfolio
Calculating Delta 1/2

• In principle calculating delta is straightforward: e.g., for *credit number i* (of the 125 underlying credits)
  ✓ calculate the tranche spread using the existing CDS spreads (base case)
  ✓ change the CDS spread of credit *i* by (say) 10 bps.
  ✓ calculate the MTM change in the value of
    ➢ CDO tranche
    ➢ the underlying CDS
  and take the ratio
Calculating Delta 2/2

• **Problem**: Monte Carlo method produces values that are *subject to error*
  ✓ Problematic to calculate change in value using *separate* MC runs – change will contain two (independent) errors and will tend to be inaccurate

• **Solution**:
  ✓ compute both base case value and revised value using *same set of random numbers* (e.g., in same MC run)
  ✓ MC errors in valuation will cancel out and estimate delta will be quite accurate (known as control variate technique)
Tranche Deltas

Delta as a Function of Subordination (Discrete CDO)

Source: Merrill Lynch

Delta as a Function of Subordination (Continuous CDO)

Source: Merrill Lynch

Acharya & Schaefer: Copula approach to valuing CDO tranches
Summary

• Important new market for credit risk transfer
  ✔ new indices high liquidity
  ✔ high rate of innovation in contract design
• Correlation a major area for future research